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On k-Optimum Dipath Partitions and Partial k-Colourings of Acyclic Digraphs

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For an acyclic digraph G and positive integer k, we give a min–max equality for:

$$\text{minimum}\{k|\mathcal{P}|+|V(G)-\bigcup\mathcal{P}|: \mathcal{P} \text{ is a node-disjoint set of dipaths in } G\}.$$

We prove that there exist k independent sets $\{S_j: j=1,2,\ldots,k\}$ such that for every optimum $\mathcal{P}$, every $P\in\mathcal{P}$ intersects each S_j, and every dipath P in $G-\bigcup\mathcal{P}$ intersects $|P|$ different sets S_j.

Given a digraph G, we let $V(G)$ denote its node-set and $E(G)$ its edge-set. By a *dipath* P in G, we mean a simple, directed path; that is, $P=(v_1, v_2, \ldots, v_m)$, $m\geq 1$, is a sequence of distinct nodes v_i such that $(v_i, v_{i+1})\in E(G)$ for $i=1,2,\ldots,m-1$. $|P|$ denotes the number of nodes in P.

For a positive integer k, a *k-independent set sequence* $(S_j: j=1,2,\ldots,k)$ in G is a sequence of k pairwise disjoint sets S_j of nodes such that if $u\in S_i$, $v\in S_j$, and $(u,v)\in E(G)$, then $i>j$. In particular, each S_j is an independent set.

Digraph G is called *acyclic* if it has no dicircuits (directed circuits, directed cycles).

In [4] and [5], the following result in proved.

(1) A MIN–MAX RELATION FOR DICIRCUITS IN A DIGRAPH. *Given any digraph H, and any fixed integer-valued $d=(d_v: v\in V(H))$, then for any fixed upper bounds $a=(a_v: v\in V(H))$ and lower bounds $b=(b_v: v\in V(H))$ on the variables $x=(x_v: v\in V(H))$ where a_v, b_v are integers or $\pm\infty$, and for any fixed integral objective coefficients $w=(w_v: v\in V(H))$, both the following linear program* (1.1) *and its dual have integer-valued optimum solutions, assuming either has an optimum solution.*

(1.1) maximize $\sum_{v\in V(H)} w_v x_v$ subject to

(1.2) $\forall$ dicircuit C of H, $\sum_{v\in C} x_v \leq \sum_{v\in C} d_v$; and

(1.3) $\forall v\in V(H)$, $b_v\leq x_v\leq a_v$.

That is, the system of inequalities given by (1.2) is box totally dual integral [8].

By linear programming duality theory, the optimum objective values of (1.1) and its dual are equal, assuming either of these exists.

Here we are interested in the following special case of (1).

(2) A MIN–MAX RELATION FOR DIPATHS IN AN ACYCLIC DIGRAPH. *Let G be an acyclic digraph and let k be a positive integer. Then the following linear program* (2.1) *and its dual* (2.4) *have integer-valued optimum solutions.*

(2.1) maximize $\sum_{v\in V(G)} z_v$ subject to

(2.2) $\forall$ dipath P of G, $\sum_{v\in P} z_v\leq k$; and

(2.3) $\forall v\in V(G)$, $z_v\leq 1$.

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(2.4) minimize $k\sum_{\text{dipaths } P} y_P + \sum_{v\in V(G)} y_v$ subject to

(2.5) $\forall v\in V(G)$, $\sum_{v\in P} y_P + y_v = 1$;

(2.6) $\forall$ dipath P of G, $y_P \geq 0$; and

(2.7) $\forall v\in V(G), y_v \geq 0$.

In other words, where $\gamma_k(G)$ *is the objective value of an optimum integer-valued solution of* (2.1), *and*

$\beta_k(G) \equiv \text{minimum}\{k|\mathcal{P}| + |V(G) - \bigcup \mathcal{P}|$: $\mathcal{P}$ is a node-disjoint set of dipaths in $G\}$,

then

(2.8) $\gamma_k(G) = \beta_k(G)$.

To obtain l.p. (2.1) from l.p. (1.1), let H be digraph G together with a new node t, and an edge from each node of G to t, and one from t to each node of G. Let

$$w_v = \begin{cases} 1 & v\in V(G) \\ 0 & v = t \end{cases}, \qquad d_v = \begin{cases} 0 & v\in V(G) \\ k & v = t \end{cases},$$

$$b_v = \begin{cases} -\infty & v\in V(G) \\ 0 & v = t \end{cases}, \qquad a_v = \begin{cases} 1 & v\in V(G) \\ 0 & v = t \end{cases}.$$

The purpose of this paper is to note interesting special structure of integer-valued optimum solutions z.

A node-disjoint set $\mathcal{P}$ of dipaths is called *k-optimum* if $k|\mathcal{P}| + |V(G) - \bigcup \mathcal{P}| = \beta_k(G)$.

(3) A PARTIAL k-COLOURING THEOREM. *Let G be an acyclic digraph. For any integer-valued optimum solution $z = (z_v : v\in V(G))$ of* (2.1), *$S(z) \equiv \{v\in V(G)): z_v = 1\}$ can be partitioned into a k-independent set sequence $(S_j : j = 1, 2, \dots, k)$, $S(z) = \bigcup_{j=1}^{k} S_j$, such that for any k-optimum set $\mathcal{P}$ of dipaths, every $P\in\mathcal{P}$ intersects each S_j and every dipath P in $G - \bigcup\mathcal{P}$ intersects $|P|$ different sets S_j.*

PROOF OF (3). Let z be an integer-valued optimum solution of (2.1). For dipath P in G, let $z(P) \equiv \sum_{v\in P} z_v$. For $v\in V(G)$ such that $z_v = 1$, define $q_v \equiv \text{maximum}\{z(P)$: P is a dipath which starts at $v\}$.

Clearly, $1 \leq q_v \leq k$, and where for $j = 1, 2, \dots, k$, we define $S_j \equiv \{v\in V(G)$: $q_v = j$, $z_v = 1\}$, $S(z) = \bigcup_{j=1}^{k} S_j$.

(4) Note that if $u\in S_i$, $v\in S_j$, and $(u, v)\in E(G)$, $i = q_u \geq z_u + q_v = l + j$. So $(S_j : j = 1, 2, \dots, k)$ is a k-independent set sequence.

Let $\mathcal{P}$ be a k-optimum set of dipaths.

Let P be any dipath of $G - \bigcup\mathcal{P}$. By complementary slackness, $z_v = 1$ $\forall v\in P$, and hence by (4), each $v\in P$ is in a different S_j.

Now consider $P\in\mathcal{P}$. By complementary slackness, $z(P) = k$. Thus, since each z_v is an integer ≤ 1, there is for each $j = 1, 2, \dots, k$, a last node, say $u(j)$, of P such that where P_j is the subpath from $u(j)$ to the end of P, we have $z(P_j) = j$. Clearly $z_{u(j)} = 1$. There is no dipath P_j' starting at $u(j)$ with $z(P_j') > j$, because then we would have $z(P') > k$ for the dipath P' obtained from P by replacing P_j with P_j'. Thus $q_{u(j)} = j$, and so $u(j)\in S_j$. This completes the proof of (3).

Regardless of whether G acyclic, for any integer-valued solution z of (2.2) and (2.3), the subgraph of G induced by $S(z)$, denoted $G_{S(z)}$, contains no dipath P with $|P| > k$.

So where we define $\alpha_k(G) \equiv \text{maximum}\{|S|: G_S$ contains no dipath with more than k nodes$\}$, and as before,

$$\gamma_k(G) \equiv \text{the objective value of an optimum integer-valued solution of (2.1),}$$

we have

(5) $\gamma_k(G) \leqslant \alpha_k(G)$.

Hence as a corollary of (2) we have: *For any acyclic digraph G,*

(6) $\beta_k(G) \leqslant \alpha_k(G)$.

In my thesis, I conjectured (6) for any digraph. However, Claude Berge pointed out to me that (6) does not hold for hypotraceable digraphs and $k = |V(G)| - 2$. G is *hypotraceable* means G does not have a Hamiltonian dipath, but $\forall v \in V(G)$, $G - v$ does have a Hamiltonian dipath. Grötschel, Thomassen, and Wakabayashi [14] have shown that there exists a hypotraceable digraph on n nodes if and only if $n \geqslant 7$.

(2) does not hold either for hypotraceable digraphs and $k = |V(G)| - 2$, since if it did, it would imply that (6) holds in this case.

(3) does not hold in this case either: For each $v \in V(G)$, where P_v is a dipath in $G - v$ with $|P_v| = k+1$, $\mathcal{P}_v \equiv \{P_v\}$ is k-optimal. However, there is no collection of k disjoint subsets S_j of $V(G)$ at all which intersect each $\mathcal{P}_v$ as in (3).

For related work see [2], [7], [10], [11], [12], [13], [15], [16], [17], and [18]. In particular, [2] and [16] contain conjectures, the acyclic cases of which are implied by (3).

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